Exercise Sheet 5

For this exercise sheet we will need the following theorems:

Cauchy's Formula: Let $f\colon \Omega \to \mathbb{C}$ be holomorphic, then for every $z_0 \in \Omega$ and every integer $n \geq 0$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

Here γ is a simple closed contour around z_0 with counter-clockwise orientation, such that its interior contained in Ω and $f^{(n)}$ is the n-th derivative ($f^{(0)}(z) = f(z)$).

Taylor Expansion: Let $f\colon \Omega \to \mathbb{C}$, be holomorphic, then for every $z_0 \in \Omega$, there exists $r > 0$, such that $D_r(z_0) \subset \Omega$ and the Taylor series of f at z_0 converge to f in $D_r(z_0)$ (i.e. every holomorphic function is analytic). The Taylor expansion at z_0 is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Liouville's Theorem: Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function, if $|f|$ is bounded then f is constant.

Uniqueness Theorem: Let $f\colon \Omega \to \mathbb{C}$ be holomorphic and let $Z = \{z \in \Omega \mid f(z) = 0\}$ be the zero set of f. If Z contains a non-constant sequence converging to a point in Ω, then $f = 0$.

1. In this exercise we'll prove Liouville's theorem, using Cauchy's formula. Let f be an entire function, such that $|f(z)| \leq M$, for all $z \in \mathbb{C}$.
   - Using Cauchy's formula compute $\int_{\gamma_R} \frac{f(z)dz}{(z-a)(z-b)}$, where $a, b \in \mathbb{C}$ are two distinct points and γ_R is the circle around 0 or radius R, $R > \max(|a|, |b|)$.
   - Not using the above item show that $\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)dz}{(z-a)(z-b)} = 0$.
   - Combine the two results to get Liouville's theorem.
2. Prove that if $f\colon \mathbb{C} \to \mathbb{C}$ is entire and $\mathrm{Re}(f)$ is bounded then f is constant.
3. Prove that if f is an entire function and f does not take values on $\mathbb{R}_{\leq 0}$, then f is constant (Hint: Take a branch of the square root function, $g(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$, it is defined and holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. What is the image of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ under g?).
4. Throughout this question n is a positive integer. The properties described hold for every n.
   - Show that there is no holomorphic function in the disk which satisfies that $f(1/n) = \frac{(-1)^n}{n^2}$.
   - Suppose f is holomorphic in the open unit disk and we know that $f(1/n) = 1/(n+1)$. What is $f(i/2)$?
   - Suppose f is entire and $f(n) = 0$ for all n. Is f necessarily the 0 function? If so, prove it. If not, find a counterexample and explain why it doesn't contradict the uniqueness theorem.

5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that there exists an integer $n > 0$ and some $R > 0$, such that $|f(z)| \leq |z|^n$, for all $z \in \mathbb{C}$, such that $|z| > R$. Prove that f is a polynomial (Hint: Write f as a Taylor series at 0 and use question 7 to deduce its radius of convergence. Using Cauchy's formula show that the coefficients of z^m, for $m > n$ vanish).

6. Let $a \in \mathbb{C}$, define $z^a = e^{a\log(z)}$, where $\log(z)$ is the branch of the logarithm analytic in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (we take $\log(z) = \log|z| + i\arg(z)$, where we only let $\arg(z)$ to take values in $(-\pi, \pi)$). Show that $((1+z)^a)' = a(1+z)^{a-1}$, using this find the coefficients of the Taylor series of $f(z) = (1+z)^a$ at 0 and the convergence radius of the expansion.

7. Let $f\colon \Omega \to \mathbb{C}$, be holomorphic. Let $a \in \Omega$ be an arbitrary point. We will prove that the Taylor series of f around a converges in the largest open disc centered at a that is contained in Ω.

   - (Non-obligatory) You will this fact in the following item. Show that given a simple closed contour γ and a function φ continuous on γ, the functions:
   $$F_n(z) = \int_\gamma \frac{\varphi(w)dw}{(w-z)^n}$$
   are holomorphic on the interior of γ. Furthermore $F_n'(z) = nF_{n+1}(z)$, for all z in the interior of γ (Hint: take a point a in the interior of γ, for z in a disc sufficiently small around a, what can you say about $F_1(z) - F_1(a)$? What happens for a general n?).
   - Let γ be a circle of radius R around a ,such that $\overline{D}_R(a) \subset \Omega$. Show that:
   $$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(z-a)^n + f_{N+1}(z)(z-a)^{N+1}.$$
   Here we have that:
   $$f_{N+1}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)dw}{(w-a)^{N+1}(w-z)}.$$
   - Show that $|f_{N+1}(z)(z-a)^{N+1}|$ tends to 0 uniformly in every disk $\overline{D}_r(a) \subset D_R(a)$.
   - Note that R can be chosen arbitrarily close to the shortest distance between a and the boundary of Ω. Deduce the original claim.

8. Using Cauchy's formula, compute the following real integrals.

   - $\int_0^{2\pi} \frac{dx}{5+\cos(x)}$,
   - $\int_0^{\pi} \frac{\cos^2(x)dx}{1-a\sin^2(x)}$, $0 < a < 1$,
   - $\int_0^{\pi} \frac{dx}{(a+b\cos(x))^2}$, $0 < b < a$.

9. Let f and g be entire functions show that $f^2(z) + g^2(z) = 1$, for every $z \in \mathbb{C}$ if and only if there exist an entire function F, such that $f(z) = \cos(F(z))$ and $g(z) = \sin(F(z))$ (Hint: show that there exists an entire function H, such that $e^H = f + ig$).

10. Let $f(z) = \frac{1}{1+z^2}$:

    - Show that $f(z)$ has a primitive in $\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} \mid z = it \text{ or } z = -it\ ,\ t \geq 1\}$. Find the primitive.

- Let F be a primitive of f in Ω satisfying $F(0) = 0$. Show that $F(\tan(z)) = z$, for all $z \in U = \{w \in \mathbb{C} \mid \tan(w) \in \Omega\}$. Here we define $\tan(z) = \frac{\sin(z)}{\cos(z)}$.

11. Let $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} z^{3n}$. Find the radius of convergence of the series, i.e. find where f is defined. Find a function F holomorphic in some domain properly containing the open disc of convergence of the series, such that $F(z) = f(z)$, whenever f is defined (Hint: what can you say about $(zf(z))'$? The function you'll get will be holomorphic on the entire plane without several lines).